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CARTESIAN CLOSED HULL OF THE CATEGORY OF UNIFORM SPACES

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A concrete category $\mathcal{K}$ is a CCT (cartesian closed topological) extension of the category Unif of uniform spaces if 1. $\mathcal{K}$ is cartesian closed, 2. Unif is a full, finitely productive subcategory of $\mathcal{K}$ and the forgetful functor of $\mathcal{K}$ extends that of Unif and 3. $\mathcal{K}$ has initial structures. We describe the smallest CCT extension of Unif which is called the CCT hull by H. Herrlich and L.D. Nel. The objects of the CCT hull are bornological uniform spaces, i.e. uniform spaces endowed with a collection of "bounded" sets related naturally to the uniformity; the morphisms are the uniformly continuous maps which preserve the bounded sets.

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Introduction

The category **Top** of topological spaces does not have canonical function spaces. This has inspired a theory of extensions of this category. For example, E. Spanier [15] has introduced the category of quasi-topologies as a suitable extension of **Top**. Here "suitable" is a category which retains the characteristic properties of **Top** (such categories are called top-categories) and has canonical function spaces, i.e. is cartesian closed. The least cartesian closed top-category extending a given concrete category $\mathcal{K}$ is called the CCT hull of $\mathcal{K}$ by H. Herrlich and L.D. Nel [8]. The CCT hull of the category **Top** has been investigated by Ph. Antoine [3], and was concretely described by G. Bourdaud [5]; see also A. Machado [11].

The aim of the present paper is to describe the CCT hull of the category **Unif** of uniform spaces. This is the category of so called *bornological uniform spaces.* These are uniform spaces endowed with a bornology, naturally related to the uniformity; the morphisms are uniformly continuous maps which preserve bounded sets. The hom-object Hom(A, B) of two uniform spaces A, B is their function space with the uniformity of uniform convergence and the bornology of all equicontinuous sets. We also describe the CCT hull of the category of metric spaces and uniformly continuous maps.

1. Preliminaries

1.1. By a *concrete category* we understand a category $\mathcal{K}$ equipped with a forgetful functor $U:\mathcal{K}\to\mathrm{Set}$ which is faithful and transportable (i.e. for each object A of $\mathcal{K}$, and each bijection $f: UA\to X$, X a set, there is a unique object B of $\mathcal{K}$ for which an isomorphism $\bar{f}: A\to B$ exists with $f=U\bar{f}$). The objects A of a concrete category are written as pairs $A=(X,\alpha)$ where X is the underlying set (i.e. $X=UA$) and α is a "structure". The morphisms $f:(X,\alpha)\to(Y,\beta)$ of $\mathcal{K}$ are described by the same symbols as the underlying maps $f:X\to Y$. Using this convention, we need not employ the symbol U explicitly. Moreover, we use harmless conventions like writing $X\xrightarrow{f}(Y,\beta)$ for a pair consisting of a map $f:X\to Y$ and an object (Y,β), or, saying that a map $f:(X,\alpha)\to(Y,\beta)$ is a morphism (which means that $f:X\to Y$ is a morphism from (X,α) to (Y,β)).

1.2. A *top-category* is a concrete category $\mathcal{K}$ which is

(a) *initially complete*, i.e., for each source $\{X\xrightarrow{f_i}(Y_i,\beta_i)\}_{i\in I}$ (where I is a set, or a class) in $\mathcal{K}$ there exists an initial structure α on X, which means that each $f_i:(X,\alpha)\to(Y_i,\beta_i)$ is a morphism and given an object (T,δ) and a map $h:T\to X$, then $h:(T,\delta)\to(X,\alpha)$ is a morphism iff each $f_i\cdot h:(T,\delta)\to(Y_i,\beta_i)$ is a morphism, $i\in I$;

(b) *well-fibred*, i.e., for each set X the collection of all objects A with $UA=X$ is a set and, moreover, it is a singleton set whenever card $X\leq 1$. The latter expresses the fact that a constant map is always a morphism and that there is just one void object.

Remarks. (i) Condition (a) above is equivalent to the existence of a final structure for each sink $\{(X_i,\alpha_i)\xrightarrow{f_i}Y\}_{i\in I}$. This is a structure β on Y such that each $f_i:(X_i,\alpha_i)\to(Y,\beta)$ is a morphism and given an object (T,δ) and a map $h: Y\to T$, then $h:(Y,\beta)\to(T,\delta)$ is a morphism iff each $h\cdot f_i:(X_i,\alpha_i)\to(T,\delta)$ is a morphism, $i\in I$.

(ii) Surjective morphisms $f:(X,\alpha)\to(Y,\beta)$ such that β is final with respect to the singleton sink $\{(X,\alpha)\xrightarrow{f}Y\}$ are called *quotients*.

(iii) Let (X,α) be an object of a top-category. For each set $Y\subseteq X$ we denote by α/Y the induced structure on Y, i.e., the initial structure of the singleton source of the inclusion map $\{Y\xrightarrow{\mathrm{in}}(X,\alpha)\}$.

Given a collection $\mathcal{A}$ of subsets of X, we call a structure α' the *$\mathcal{A}$-final modification* of α provided that α' is final with respect to the following sink of inclusion maps:

$$\{(Y,\alpha/Y)\xrightarrow{\mathrm{in}}X\}_{Y\in\mathcal{A}}.$$

If $\alpha = \alpha'$, then α is said to be *final with respect to* $\mathscr{A}$.

1.3. Convention. By a *subcategory* of a category $\mathscr{K}$ we shall always mean a full subcategory $\mathscr{L}$ such that, for each object A of $\mathscr{L}$, all objects, isomorphic with A in $\mathscr{K}$, are also in $\mathscr{L}$. Thus, each subcategory of a concrete category $\mathscr{K}$ is also concrete, with the forgetful functor a restriction of that of $\mathscr{K}$.

Recall that a subcategory $\mathscr{L}$ of a concrete category $\mathscr{K}$ is said to be *initially dense* if each object of $\mathscr{K}$ is initial for some source in $\mathscr{L}$; analogously, *finally dense*.

1.4. The following conditions are proved to be equivalent for each top-category $\mathscr{K}$ in [6]:

(i) $\mathscr{K}$ is cartesian closed;

(ii) For each object A of $\mathscr{K}$ the product functor $A \times - : \mathscr{K} \to \mathscr{K}$ preserves colimits;

(iii) *Coproducts are productive* (i.e., the canonical morphism

$$\coprod_{i \in I} A \times B_i \to A \times \coprod_{i \in I} B_i$$

is an isomorphism for arbitrary objects A and $B_i (i \in I)$ of $\mathscr{K}$) and *quotients are productive* (i.e., $1_A \times f : A \times B \to A \times C$ is a quotient for each object A and each quotient $f : B \to C$ in $\mathscr{K}$);

(iv) Each pair of objects $A = (X, \alpha)$ and $B = (Y, \beta)$ of $\mathscr{K}$ has a *canonical hom-object*, i.e. an object $\mathrm{Hom}(A, B)$, the underlying set of which is $\mathrm{hom}(A, B)$, with the following universal property: for any object $C = (Z, \gamma)$, a map $f : Z \times X \to Y$ is a morphism

$$f : C \times A \to B$$

iff the map $\hat{f} : Z \to \mathrm{hom}(A, B)$, defined by

$$\hat{f}(z) = f(z, -) \quad \text{for each } z \in Z$$

is a morphism

$$\hat{f} : C \to \mathrm{Hom}(A, B).$$

1.5. We are interested in CCT (*cartesian closed top-*) *extensions* of a given concrete category $\mathscr{K}$. These are defined to be CCT categories $\mathscr{L}$ such that $\mathscr{K}$ is a finitely productive subcategory of $\mathscr{L}$. Not each top-category has a CCT extension; an example can be found in [1]. But, whenever $\mathscr{K}$ has a CCT extension, then it has a smallest CCT extension, called the *CCT hull* of $\mathscr{K}$. This CCT extension $\mathscr{K}^*$, introduced by H. Herrlich and L.D. Nel [8], is characterized by any of the following equivalent conditions:

(i) For each CCT extension $\mathscr{L}$ of $\mathscr{K}$ there exists a full concrete embedding $I : \mathscr{K}^* \to \mathscr{L}$, the restriction of which to $\mathscr{K}$ is $1_{\mathscr{K}}$;

(ii) $\mathscr{K}$ is finally dense in $\mathscr{K}^*$ and the hom-objects $\mathrm{Hom}(A, B)$ with A, B in $\mathscr{K}$ are initially dense in $\mathscr{K}^*$.

The following proposition is proved in [8].

1.6. Proposition. *Let $\mathcal{K}$ be a category with finite products which is a finally dense subcategory of a CCT category $\mathcal{L}$. Let $\mathcal{K}'$ be the subcategory of $\mathcal{L}$ formed by all hom-objects* $\mathrm{Hom}(A, B)$ *with A, B in $\mathcal{K}$. Then the CCT hull of $\mathcal{K}$ is the subcategory of $\mathcal{L}$ formed by all initial objects (in $\mathcal{L}$) of sources in $\mathcal{K}'$. Hom-objects in the CCT hull are formed as those in $\mathcal{L}$.*

2. Bornological uniform spaces

2.1. Recall from [9] that a *bornology* on a set X is a collection $\mathcal{A}$ of subsets of X such that (i) each finite subset is in $\mathcal{A}$, (ii) if $P \in \mathcal{A}$, then $Q \in \mathcal{A}$ for all $Q \subset P$ and (iii) if $P, Q \in \mathcal{A}$, then $P \cup Q \in \mathcal{A}$. The elements of $\mathcal{A}$ are called *bounded* sets.

Whenever working with uniform spaces, we always use the definition via pseudometrics (see [16]).

Definition. A *bornological uniform space* is a triple $(X, \alpha, \mathcal{A})$ consisting of a uniform space (X, α) and a bornology $\mathcal{A}$ on X such that

(i) α is $\mathcal{A}$-final, i.e., α contains each pseudometric which is uniformly continuous on all bounded sets;

(ii) $\mathcal{A}$ is *α-closed*, i.e., $\mathcal{A}$ contains each set $M \subseteq X$ with the following property:

$$(*)\qquad \begin{array}{l}\text{for each pseudometric } a \text{ in } \alpha \text{ and each } \varepsilon > 0,\\ \text{there exists a bounded set } \tilde{M} \in \mathcal{A} \text{ such that}\\ \qquad a(x, \tilde{M}) < \varepsilon \quad \text{for each } x \in M.\end{array}$$

Notation. We denote by **BUnif** the category of bornological uniform spaces. Its morphisms from $(X, \alpha, \mathcal{A})$ to $(Y, \beta, \mathcal{B})$ are those uniformly continuous maps $f: (X, \alpha) \to (Y, \beta)$ which preserve bounded sets, i.e., $P \in \mathcal{A}$ implies $f(P) \in \mathcal{B}$.

We consider **Unif** as a subcategory of **BUnif** by identifying each uniform space (X, α) with the bornological uniform space $(X, \alpha, \exp X)$, where $\exp X = \{Y;\ Y \subset X\}$.

Remark. If we replace "$\tilde{M}$ bounded" by "$\tilde{M}$ finite" in the above condition $(*)$, then the sets M, satisfying this new condition, are precisely the precompact sets in (X, α). Therefore, the bornology of each bornological uniform space contains all precompact subsets.

2.2. Proposition. ***BUnif** is a CCT category.*

Remarks. (i) In the course of proof we shall see that the hom-object of two bornological uniform spaces $B = (Y, \beta, \mathcal{B})$ and $C = (Z, \gamma, \mathcal{C})$ is the following space

$$\mathrm{Hom}(B, C) = (\mathrm{hom}(B, C), \sigma, \mathcal{S}):$$

$\mathcal{S}$ consists of those sets $P \subset \hom(B, C)$ which are

(a) *equicontinuous*, i.e., for every pseudometric $c \leq 1$ in γ, the pseudometric

$$c_P(x, y) = \sup_{f \in P} c(f(x), f(y)), \quad (x, y \in Y)$$

belongs to β, and

(b) *equibounded*, i.e., for each $Q \in \mathcal{B}$ we have $\bigcup_{f \in P} f(Q) \in \mathcal{C}$;

and σ is the $\mathcal{S}$-final modification of the uniformity of uniform convergence on bounded sets.

(ii) The following assertion is easy to prove (and can be considered as a folklore): given uniform spaces A, B and C, a map $f: A \times B \to C$ is uniformly continuous iff

(a) $\hat{f}(A) \subset \hom(B, C)$ is an equicontinuous family (for $\hat{f}$ see 1.4) and

(b) $\hat{f}: A \to \hom(B, C)$ is uniformly continuous with respect to the uniformity of uniform convergence on $\hom(B, C)$.

Proof of the proposition. *1. Initial completeness of* ***BUnif***. Let $\{f_i : X \to (X_i, \alpha_i, \mathcal{A}_i); i \in I\}$ be a source in **BUnif**. Let $\mathcal{A}$ be the bornology $\{P \subset X; f_i(P) \in \mathcal{A}_i$ for every $i \in I\}$. Let α be the $\mathcal{A}$-final modification of the initial uniformity of the source $\{f_i : X \to (X_i, \alpha_i); i \in I\}$ in **Unif**. First we shall prove that $\mathcal{A}$ is α-closed. Let $M \subset X$ be a set such that for every $a \in \alpha$ and every $\varepsilon > 0$ there is $\tilde{M} \in \mathcal{A}$ with $a(x, \tilde{M}) < \varepsilon$ for $x \in M$. For every $i \in I$ and $a_i \in \alpha_i$, the pseudometric $a(x, y) = a_i(f_i(x), f_i(y))$ belongs to α. Hence, for every $\varepsilon > 0$ there exists $\tilde{M} \in \mathcal{A}$ with $a_i(f_i(x), f_i(\tilde{M})) < \varepsilon$ for all $x \in M$. As $f_i(\tilde{M}) \in \mathcal{A}_i$, it follows from the α_i-closedness of $\mathcal{A}_i$ that $f_i(M) \in \mathcal{A}_i$. Thus, $M \in \mathcal{A}$. Hence, $(X, \alpha, \mathcal{A})$ is a bornological uniform space.

Second, each $f_i : (X, \alpha, \mathcal{A}) \to (X_i, \alpha_i, \mathcal{A}_i)$ is a morphism. This follows immediately from the definition of $(X, \alpha, \mathcal{A})$.

Third, let $g: Y \to X$ be a map such that every $f_i \cdot g : (Y, \beta, \mathcal{B}) \to (X_i, \alpha_i, \mathcal{A}_i)$ is a morphism. Then $P \in \mathcal{B}$ implies $f_i(g(P)) \in \mathcal{A}_i (i \in I)$, and hence, $g(P) \in \mathcal{A}$. Also, every $f_i \cdot g$ is uniformly continuous on every $P \in \mathcal{B}$ and thus, g is uniformly continuous on P with respect to the initial uniformity on X defined by the source $\{f_i : X \to (X_i, \alpha_i); i \in I\}$. Since we know that $g(P) \in \mathcal{A}$, it follows that g is uniformly continuous on P with respect to the uniformity α on X. As (Y, β) is $\mathcal{B}$-final, we conclude that g is uniformly continuous on all of (Y, β). Thus $g: (Y, \beta, \mathcal{B}) \to (X, \alpha, \mathcal{A})$ is a morphism of **BUnif**.

2. Finite products. The product $(X, \alpha, \mathcal{A}) \times (Y, \beta, \mathcal{B})$ of two bornological uniform spaces has the underlying set $X \times Y$, its bornology $\mathcal{C}$ consists precisely of all subsets of products $P \times Q$ $(P \in \mathcal{A}, Q \in \mathcal{B})$ and its uniformity is the $\mathcal{C}$-final modification of the product uniformity $(X, \alpha) \times (Y, \beta)$. This follows from the preceding description of initial structures.

3. Hom-objects. (A) First, we shall verify that the hom-objects, as described in Remark (i), are bornological uniform spaces, i.e., that $\mathcal{S}$ is σ-closed.

Let $M \subset \hom(B, C)$ be a set such that for every pseudometric $s \in \sigma$ and every $\varepsilon > 0$ there exists an equibounded and equicontinuous set $\tilde{M} \subset \hom(B, C)$ with $s(f, \tilde{M}) < \varepsilon$ for all $f \in M$. Then we shall prove that M is equibounded and equicontinuous.

Let $Q \in \mathscr{B}$. For every pseudometric $c \in \gamma$ with $c \leq 1$, consider the pseudometric $s \in \sigma$ defined by

$$s(f, g) = \sup_{x \in Q} c(f(x), g(x)).$$

Given $\varepsilon > 0$, we have an equibounded and equicontinuous set $\tilde{M}$ such that $s(f, \tilde{M}) < \varepsilon$ for $f \in M$. Then for each $f \in M$ there exists $g_f \in \tilde{M}$ with $c(f(x), g_f(x)) < \varepsilon$ for all $x \in Q$. Hence, $c(f(x), g_f(Q)) < \varepsilon$ for all $x \in Q$. Thus, $c(f(x), \tilde{Q}) < \varepsilon$ for all $x \in Q$ and $f \in M$, where $\tilde{Q} = \bigcup_{f \in M} g_f(Q)$. As $\tilde{M}$ is equibounded, we have $\tilde{Q} \in \mathscr{C}$ and therefore, $\{f(x); f \in M, x \in Q\} \in \mathscr{C}$ by the γ-closedness of $\mathscr{C}$. Hence, M is equibounded.

Further, since $\tilde{M}$ is equicontinuous, the pseudometric

$$c_{\tilde{M}}(x, y) = \sup_{f \in \tilde{M}} c(f(x), f(y))$$

belongs to β. For $x, y \in Q$ and $f \in M$ we have

$$\begin{aligned} c(f(x), f(y)) &\leq c(f(x), g_f(x)) + c(g_f(x), g_f(y)) + c(g_f(y), f(y)) \\ &< \varepsilon \qquad\qquad\qquad + c_{\tilde{M}}(x, y) \qquad\qquad + \varepsilon. \end{aligned}$$

It follows that $c_M(x, y) = \sup_{f \in M} c(f(x), f(y)) \leq c_{\tilde{M}}(x, y) + 2\varepsilon$ for $x, y \in Q$. Since neither $c_M(x, y)$ nor $c_{\tilde{M}}(x, y)$ depends on ε, we have $c_M(x, y) \leq c_{\tilde{M}}(x, y)$ for $x, y \in Q$. Hence, c_M is uniformly continuous on Q. Since β is $\mathscr{B}$-final, this implies $c_M \in \beta$, i.e., M is equicontinuous.

(B) We shall prove that for arbitrary bornological uniform spaces $A = (X, \alpha, \mathscr{A})$, $B = (Y, \beta, \mathscr{B})$, $C = (Z, \gamma, \mathscr{C})$, a map $f: A \times B \to C$ is a morphism iff $\hat{f}$ (defined by $\hat{f}(a) = f(a, \text{-}))$ is a morphism, $f: A \to \operatorname{Hom}(B, C)$, see 1.4(iv).

First, let $f: A \times B \to C$ be a morphism. Then for arbitrary $P \in \mathscr{A}$ and for every $Q \in \mathscr{B}$ we have $f(P \times Q) \in \mathscr{C}$. Hence, $\bigcup_{p \in P} \hat{f}(p)(Q) = f(P \times Q) \in \mathscr{C}$, i.e., $\hat{f}(P)$ is equibounded. Also by Remark (ii) above, the uniform continuity of $f: P \times Q \to C$ implies the equicontinuity of $\hat{f}(P)$ on Q. Since β is $\mathscr{B}$-final, $\hat{f}(P)$ is equicontinuous. Hence, $\hat{f}: A \to \operatorname{Hom}(B, C)$ preserves bounded sets. Further, the uniform continuity of $f: P \times Q \to C$ for all $Q \in \mathscr{B}$ implies the uniform continuity of $\hat{f}: P \to \hom(B, C)$ with respect to the uniformity of uniform convergence on bounded sets on $\hom(B, C)$, and therefore with respect to σ, because $\hat{f}(P) \in \mathscr{S}$. Hence, $\hat{f}: A \to \operatorname{Hom}(B, C)$ is a morphism.

Second, let $\hat{f}: A \to \operatorname{Hom}(B, C)$ be a morphism. Then for every $P \in \mathscr{A}$, the set $\hat{f}(P)$ is equibounded, i.e., $f(P \times Q) = \bigcup_{p \in P} \hat{f}(p)(Q) \in \mathscr{C}$ for all $Q \in \mathscr{B}$, and hence, f preserves bounded sets. Also, $\hat{f}(P)$ is equicontinuous and $\hat{f}: P \to \hom(B, C)$ is uniformly continuous with respect to the uniformity of uniform convergence on bounded sets on $\hom(B, C)$ and thus, $f: P \times Q \to C$ is uniformly continuous. Hence, $f: A \times B \to C$ is a morphism. The proof is finished.

2.3. The key of the proof of our main result below is the fact that each bornological uniform space is initial for some source of hom-objects of uniform spaces. We formulate this as a separate lemma.

A bornological uniform space $(X, \alpha, \mathscr{A})$ is said to be *metrizable* if there exists a metric d such that α is the $\mathscr{A}$-final modification of the uniformity induced by d. A *subspace* of a bornological uniform space $(X, \alpha, \mathscr{A})$ is a space $(Y, \beta, \mathscr{B})$ where $Y \subset X$, $\mathscr{B} = \mathscr{A} \cap \exp Y$ and β is the $\mathscr{B}$-final modification of α / Y.

We denote by I the segment $\langle 0, 1\rangle$ with its usual uniformity.

Lemma. *Each metrizable bornological uniform space A can be embedded into* $\mathrm{Hom}(A^*, I)$ *for some uniform space A^*.*

Proof. Let $A = (X, \alpha, \mathscr{A})$ and let d be a metric such that $d \leqslant 1$ and α is the $\mathscr{A}$-final modification of the uniformity induced by d. Each bounded set $M \in \mathscr{A}$ defines a pseudometric d_M on $\exp X - \{\emptyset\}$ (the set of all non-empty subsets of X), where

$$d_M(T, S) = 0 \quad \text{if } T = S \text{ or } T \cap S \supset M$$

and

$$d_M(T, S) = 1 \quad \text{otherwise.}$$

Note that $d_{M_1 \cup M_2} = \max(d_{M_1}, d_{M_2})$ and thus, the collection $\{d_M;\, M \in \mathscr{A}\}$ is a base of a uniformity γ on $\exp X - \{\emptyset\}$. Put $A^* = (\exp X - \{\emptyset\}, \gamma)$. For each $x \in X$, the distance map

$$e(x) : \exp X - \{\emptyset\} \to I$$

defined by

$$e(x)(T) = d(x, T)$$

is uniformly continuous from A^* to I. Indeed, for every $\varepsilon > 0$ and for $\delta = 1/2$,

$$d_{\{x\}}(T, S) < \delta \quad \text{implies} \quad |e(x)(T) - e(x)(S)| < \varepsilon.$$

Thus, we get a one-to-one map

$$e : X \to \mathrm{hom}(A^*, I).$$

We shall prove that e is an isomorphism of A onto the subspace of $\mathrm{Hom}(A^*, I)$ whose underlying set is $e(X)$.

By Remark (i) above, $\mathrm{Hom}(A^*, I) = \mathrm{hom}((A^*, I), \sigma, \mathscr{S})$ where $\mathscr{S}$ is the collection of all equicontinuous subsets of $\mathrm{hom}(A^*, I)$ and σ is the $\mathscr{S}$-final modification of the uniformity of uniform convergence. Thus, $\mathrm{Hom}(A^*, I)$ is metrizable by the following metric $\hat{d}$ on $\mathrm{hom}(A^*, I)$:

$$\hat{d}(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

To prove that e is an isomorphism, it suffices to verify the following:

(i) A set $M \subseteq X$ is bounded iff $e(M)$ is equicontinuous;

(ii) α is the initial uniformity of $\{X \xrightarrow{e} (\mathrm{hom}(A^*, I), \sigma)\}$.

Ad.(i) If $M \in \mathscr{A}$, then the pseudometric d_M has the following property: for each map $d(x, -) \in e(M)$ where $x \in M$,

$$d_M(T, S) < \tfrac{1}{2} \quad \text{implies } T = S \text{ or } T \cap S \supset M$$

and therefore, $d(x, T) - d(x, S) = 0$. Hence, $e(M)$ is equicontinuous.

Conversely, if $e(M)$ is equicontinuous, we shall prove that M has the property $(*)$ of 2.1. For each $\varepsilon > 0$, the equicontinuity of $e(M)$ guarantees the existence of $\delta > 0$ and a pseudometric $d_{\tilde{M}}$ (where $\tilde{M} \in \mathscr{A}$) such that for $S, T \in \exp X - \{\emptyset\}$,

$$d_{\tilde{M}}(S, T) < \delta \quad \text{implies} \quad |e(x)(S) - e(x)(T)| < \varepsilon \quad \text{for } x \in M.$$

In particular, for $S = \tilde{M}$, $T = X$ we get

$$|e(x)(S) - e(x)(T)| = |d(x, \tilde{M}) - 0| < \varepsilon \quad \text{for } x \in M.$$

Then $M \in \mathscr{A}$ because $\mathscr{A}$ is α-closed.

Ad.(ii) Since α is $\mathscr{A}$-final, it suffices to prove that for each $P \in \mathscr{A}$ the uniformity α/P is initial for $\{P \xrightarrow{e} (e(P), \sigma/e(P))\}$. By (i) we know that $e(P)$ is an equicontinuous set, hence, $\sigma/e(P)$ is the uniformity of uniform convergence on $e(P)$. Thus, α/P is metrizable by d and $\sigma/e(P)$ is metrizable by $\hat{d}$. Since, for arbitrary $x, y \in X$,

$$d(x, y) = \hat{d}(e(x), e(y)),$$

clearly α/P is initial.

2.4. Theorem. ***BUnif*** *is the* CCT *hull of* ***Unif.***

Proof. It is clear that **Unif** is finally dense in **BUnif**: each bornological uniform space $(X, \alpha, \mathscr{A})$ is a final object (in **BUnif**) of the sink of all inclusion maps from $(P, \alpha/P, \exp P)$ into $X (P \in \mathscr{A})$.

By 1.6, we have to prove that every bornological uniform space is an initial object of a source of hom-objects of uniform spaces. Let $A = (X, \alpha, \mathscr{A})$ be a bornological uniform space. It is sufficient to prove that A is initial for a source $\{X \xrightarrow{f_j} B_j\}$ of metrizable bornological uniform spaces. (By Lemma 2.3, for each j there exists an embedding $e_j : B_j \to \mathrm{Hom}(B_j^*, I)$. Clearly, A is initial for the source

$$\{X \xrightarrow{e_j \cdot f_j} \mathrm{Hom}(B_j^*, I)\}.)$$

For each pseudometric $a \in \alpha$ denote by $\sim_a$ the equivalence on X defined by $x \sim_a y$ iff $a(x, y) = 0$; let $f_a : X \to X/\sim_a$ be the canonical map and let β_a be the uniformity on $X/\sim_a$, defined by the metric $\bar{a}(x, y) = a(f_a^{-1}(x), f_a^{-1}(y))$. Further, let $\mathscr{B}_a$ be the following bornology on $X/\sim_a$:

$$\mathscr{B}_a = \{f_a(P);\ P \in \mathscr{A}\}.$$

Then the bornology $\bar{\mathscr{B}}_a$, obtained from $\mathscr{B}_a$ by adding all M such that for each $\varepsilon > 0$ there exists $\tilde{M} \in \mathscr{B}_a$ with $a(x, \tilde{M}) < \varepsilon$ for all $x \in M$, is obviously β_a-closed and so it is $\bar{\beta}_a$-closed where $\bar{\beta}_a$ is the $\bar{\mathscr{B}}_a$-final modification of β_a. Put $B_a = (X/{\sim_a}, \bar{\beta}_a, \bar{\mathscr{B}}_a)$. Then A is initial for the source

$$\{X \xrightarrow{f_a} B_a\}_{a \in \alpha}.$$

Indeed, $P \in \mathscr{A}$ implies $f_a(P) \in \bar{\mathscr{B}}_a$. Conversely, if $f_a(P) \in \bar{\mathscr{B}}_a$ for each $a \in \alpha$, then $P \in \mathscr{A}$ (because P has the property (*) of 2.1). The uniformity α is clearly initial (in **Unif**) for the source

$$\{X \xrightarrow{f_a} (X/{\sim_a}, \beta_a)\}_{a \in \alpha}.$$

Since, moreover, α is $\mathscr{A}$-final and, for each $P \in \mathscr{A}$, the uniformities β_a and $\bar{\beta}_a$ coincide on $f_a(P)$, it is clear that α is also initial for the source

$$\{X \xrightarrow{f_a} (X/{\sim_a}, \bar{\beta}_a)\}_{a \in \alpha}.$$

3. Bornological metrically generated spaces

3.1. We are going to describe the CCT hull of the category Met of metric spaces and uniformly continuous maps. Actually, we consider two metric spaces identical if they induce the same uniformity, thus, Met is, more precisely, the category of metrizable uniform spaces. (Otherwise, we would lose transportability, see 1.1.)

The coreflective hull of **Met** in **Unif** is the category **Metg** of metrically generated uniform spaces. These are quotients (in **Unif**) of sums of metric spaces. We are going to prove that **Met** and **Metg** have the same CCT hull. The advantage of **Metg** is that it is a top-category.

3.2. Definition. A *bornological metrically generated space* is a triple $(X, \alpha, \mathscr{A})$ consisting of a metrically generated uniform space (X, α) and a bornology $\mathscr{A}$ on X such that

(i) α is $\mathscr{A}$-final (in **Metg**);

(ii) $\mathscr{A}$ is strongly α-closed, i.e., $\mathscr{A}$ contains each set $M \subseteq X$ with the following property:

(*) for each pseudometric a in α, each $\varepsilon > 0$ and each infinite set $M_1 \subseteq M$ there exists a bounded set $\tilde{M}_1 \in \mathscr{A}$ such that for infinitely many points $x \in M_1$ we have $a(x, \tilde{M}_1) < \varepsilon$.

Notation. We denote by **BMet** the category of bornological metrically generated spaces. Its morphisms are uniformly continuous maps preserving bounded sets.

Metg and **Met** are considered as subcategories of **BMet** via the trivial bornology (in which each set is bounded).

3.3. Theorem. *BMet is the* CCT *hull of Met.*

Proof. Denote by $\mathscr{L}$ the subcategory of **BUnif** consisting of all $(X, \alpha, \mathscr{A})$ such that (X, α) is in **Metg** and α is $\mathscr{A}$-final in **Metg.** Then $\mathscr{L}$ is coreflective in **BUnif**: the $\mathscr{L}$-coreflection of a bornological uniform space $(X, \alpha, \mathscr{A})$ is $(X, \alpha', \mathscr{A})$, where α' is the final uniformity of the sink of inclusion maps $\{P \to X; P \in \mathscr{A}\}$, where P is equipped with the **Metg**-coreflection of the subspace uniformity. Further, $\mathscr{L}$ is closed in **BUnif** under finite products; this follows from the description of finite products in **BUnif** (see 2.2) and from the fact that the **Metg** is closed in **Unif** under finite products [10]. It follows that $\mathscr{L}$ is a CCT category, and initial objects and hom-objects in $\mathscr{L}$ are $\mathscr{L}$-coreflections of those in **BUnif**, see [12].

Met is finally dense in $\mathscr{L}$: each object $(X, \alpha, \mathscr{A})$ in $\mathscr{L}$ is a final object of the sink

$$\{(P, \alpha /\!/ P, \exp P) \xrightarrow{\text{in}} X\}_{P \in \mathscr{A}}$$

of inclusion maps, where $\alpha /\!/ P$ means the **Metg**-coreflection of the subspace uniformity on P. Each $(P, \alpha /\!/ P, \exp P)$ is a final object of a sink

$$\{(X_i, \alpha_i, \exp X_i) \xrightarrow{f_i} P\}_{i \in I} \cup \{(P, \delta, \exp P) \xrightarrow{\text{id}} P\}$$

where δ is the (metrizable!) discrete uniformity and

$$\{(X_i, \alpha_i) \xrightarrow{f_i} P\}_{i \in I}$$

is a sink in **Metg** whose final object in **Metg** is $(P, \alpha /\!/ P)$.

Using Proposition 1.6, we shall prove the theorem by verifying that an object A of $\mathscr{L}$ is in **BMet** iff A is an initial object (in $\mathscr{L}$) for some source of hom-objects of metric spaces.

1) First, for each bornological metrically generated space $A = (X, \alpha, \mathscr{A})$ we shall exhibit a source of hom-objects of metric spaces, the initial object of which is A. To this end, we use the fact that **Met** is initially dense in **Unif**; thus, there exists a source

$$\{X \xrightarrow{h_i} H_i\}_{i \in I}$$

of metric spaces H_i such that (X, α) is its initial object in **Unif** and, consequently, in **Metg.** We are going to enlarge this source to "catch" the bornology $\mathscr{A}$.

Consider $I = \langle 0, 1 \rangle$ and $N = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ as subspaces of the real line with the usual metric. Recall from 2.2, Remark (i) that, in **BUnif**,

$$\text{Hom}(N, I) = (\text{hom}(N, I), \sigma, \mathscr{S}),$$

where $\mathscr{S}$ is the collection of all equicontinuous sets and σ is the $\mathscr{S}$-final modification of the uniformity of uniform convergence on hom(N, I). Since the latter is metrizable by the metric

$$d(f, g) = \sup\{|f(1/k) - g(1/k)|;\ k = 1, 2, 3, \ldots\},$$

$(\hom(N, I), \sigma, \mathscr{S})$ is in $\mathscr{L}$ and serves as a hom-object of N and I in $\mathscr{L}$.

For each non-void set $M \in \exp X - \mathscr{A}$ we use the fact that M does not satisfy (*) of 3.2: we choose a pseudometric $a \in \alpha$, a number $\varepsilon > 0$ and an infinite subset $M_1 \subset M$ such that

$$M_1 \cap \{x \in X;\ a(x, R) < \varepsilon\}$$

is a finite set for each $R \in \mathscr{A}$. Of course, a can be chosen in such a way that $a \leq 1$ and, without loss of generality, $\varepsilon = 1$. Moreover, since M is not precompact (see Remark 2.1), it contains a countable infinite uniformly discrete subset. Replacing M_1 by this subset, we can assume that

$$M_1 = \{m_1, m_2, m_3, \ldots\}$$

where $a(m_i, m_j) = 1$ whenever $i \neq j$. Then for each $R \in \mathscr{A}$ there exists k_0 such that

$$x \in R \text{ and } k \geq k_0 \text{ imply } a(x, \{m_k, m_{k+1}, m_{k+2}, \ldots\}) = 1.$$

For each $x \in X$ denote by $f_x : N \to I$ the map defined by

$$f_x(1/k) = a(x, \{m_k, m_{k+1}, m_{k+2}, \ldots\}).$$

Then f is uniformly continuous, in fact, for each set $R \in \mathscr{A}$ the collection $\{f_x;\ x \in R\}$ is equicontinuous. All this depends on the non-empty set $M \in \exp X - \mathscr{A}$; define a map

$$p_M : X \to \hom(N, I)$$

by

$$p_M(x) = f_x.$$

This map preserves bounded sets: $R \in \mathscr{A}$ implies $p_M(R) \in \mathscr{S}$. It is also uniformly continuous, in fact, a contraction: for each $x, x' \in X$ we have

$$d(p_M(x), p_M(x')) = \sup|f_x(1/k) - f_{x'}(1/k)| \leq a(x, x').$$

Thus, $p_M : A \to \operatorname{Hom}(N, I)$ is a morphism in **BMet**.

We claim that A is the initial object of the following source:

$$\{X \xrightarrow{h_i} H_i\}_{i \in I} \cup \{X \xrightarrow{p_M} \operatorname{Hom}(N, I)\}_{M \in \exp X - \mathscr{A}}.$$

Indeed, each h_i and p_M is a morphism from A. The first subsource determines the uniformity of A. Thus, if suffices to check that, given $R \subseteq X$,

$$R \in \mathscr{A} \text{ iff } p_M(R) \text{ is equicontinuous for each } M \in \exp X - \mathscr{A}.$$

This is obvious: we have observed already that $R \in \mathscr{A}$ implies that $p_M(R)$ is equicontinuous for all $M \in \exp X - \mathscr{A}$; if $R \notin \mathscr{A}$ then $p_R(R)$ fails to be equicontinuous since for each k,

$$|p_R(m_k)(1/k+1)-p_R(m_k)(1/k)|=1.$$

Thus, given $\varepsilon>0$, any $k>1/\delta$ has the property that $|1/k-1/(k+1)|<\delta$ yet, the values of $p_R(m_k)\leqslant p_R(R)$ at $1/k$ and $1/(k+1)$ have distance 1.

2) Conversely, let A be an initial object of a source of hom-objects of metric spaces. To prove that A is in **BMet**, it suffices to show that **BMet** is closed under initial objects in $\mathscr{L}$ and contains all hom-objects of metric spaces.

(a) Initial objects. Let $(X, \alpha, \mathscr{A})$ be the initial object (in $\mathscr{L}$) of a source

$$\{X \xrightarrow{f_i} (Y_i, \beta_i, \mathscr{B}_i)\}_{i\in I}$$

of bornological metrically generated spaces. We shall prove that $\mathscr{A}$ is strongly α-closed. Let $M\subset X$ have the property $(*)$ of 3.2. To prove that $M\in\mathscr{A}$, it suffices to verify that, for each $i\in I$, also $f_i(M)$ has the property $(*)$ with respect to $(Y_i, \beta_i, \mathscr{B}_i)$.

Let $b\in\beta_i$, $\varepsilon>0$ and $M_1\subseteq f_i(M)$ be given, with M_1 infinite. Choose a set $M_2\subseteq M$ such that $M_1=f_i(M_2)$ and f_i is one-to-one on M_2; define a pseudometric a on X by $a(x,x')=b(f_i(x),f_i(x'))$-clearly $a\in\alpha$. Since M has the property $(*)$, there exists $\tilde{M}_2\in\mathscr{A}$ such that $a(x,\tilde{M}_2)<\varepsilon$ for infinitely many $x\in M_2$. Then the set $\tilde{M}_1=f_i(\tilde{M}_2)\in\mathscr{B}_i$ has the property that $b(f_i(x),\tilde{M}_1)<\varepsilon$ for infinitely many $f_i(x)\in M_1$. This proves that $f_i(M)\in\mathscr{B}_i$ for each $i\in I$. Thus, $M\in\mathscr{A}$.

(b) Hom-objects. Let A, B be two metric spaces, $A=(X,\alpha)$, $B=(Y,\beta)$, where α and β are induced by metrics a and b, respectively.

We have $\mathrm{Hom}(A,B)=(\mathrm{hom}(A,B),\sigma,\mathscr{S})$ where $\mathscr{S}$ is the set of all equicontinuous sets and σ is the $\mathscr{S}$-final modification of the (metrizable!) uniformity of uniform convergence. So we have to prove that any set $M\subseteq\mathrm{hom}(A,B)$ satisfying $(*)$ of 3.2 is equicontinuous; that is, for each $\varepsilon>0$, we are to exhibit $\delta>0$ such that $a(x,x')<\delta$ implies $b(f(x),f(x'))<\varepsilon$ for all $f\in M$. Assume that, to the contrary, for each $\delta=1/n$ $(n=1,2,3\ldots)$ there exist $x_n, x'_n\in X$ and $f_n\in M$ such that $a(x,x')<1/n$, yet, $b(f_n(x_n),f_n(x'_n))\geqslant\varepsilon$. Then $M'=\{f_1,f_2,f_3,\ldots\}$ is an infinite set (indeed, each f_k is uniformly continuous, hence, $f_n=f_k$ cannot hold for infinitely many k's). We can assume $b\leqslant 1$ and then we denote by d the usual metric inducing the uniformity of uniform convergence,

$$d(f,g)=\sup_{x\in X} b(f(x),g(x)).$$

By virtue of $(*)$ in 3.2, there exists an equicontinuous set $M_1\subseteq\mathrm{hom}(A,B)$ such that $d(f_n,M_1)<\varepsilon/3$ holds for infinitely many indices n; thus, for each of these n we have a $g_n\in M_1$ with $d(f_n,g_n)<\varepsilon/3$. Then $b(g_n(x_n),g_n(x'_n))\geqslant b(f_n(x_n),f_n(x'_n))-b(f_n(x_n),g_n(x_n))-b(f_n(x'_n),g_n(x'_n))>\varepsilon/3$. Then the set of these $g_n\in M_1$ is not equicontinuous, in contradiction to the equicontinuity of M_1. This concludes the proof of the theorem.

4. A general construction

4.1. The basic property of the category **Unif** which makes a construction of a CCT extension of the above type possible is that quotients are productive (see 1.4.(iii)). More in general, we present a construction of a CCT extension for each top-category with productive quotients. It follows that such a category always has a CCT hull (which need not be the case for an arbitrary top-category, see [1]). A similar construction sketched by L.D. Nel [13] seems unfortunately not to be quite correct.

Convention. Let (X, α) and (Y, β) be objects of a concrete category. Put

$$(X, \alpha) \sqsubseteq (Y, \beta)$$

if $X \subseteq Y$ and the inclusion map $\mathrm{in}: X \to Y$ is a morphism, $\mathrm{in}: (X, \alpha) \to (Y, \beta)$. In case $X = Y$, α is said to be finer than β.

4.2. Construction. Let $\mathcal{K}$ be a top-category with productive quotients. Denote by $A\mathcal{K}$ the following concrete category.

Objects are pairs $A = (X, \mathfrak{A})$ where X is a set and $\mathfrak{A}$ is a collection of objects of $\mathcal{K}$ such that

(i) the underlying set of each $A' \in \mathfrak{A}$ is a subset of X;

(ii) if $A' \in \mathfrak{A}$ and $A'' \sqsubseteq A'$, then $A'' \in \mathfrak{A}$;

(iii) each object with an underlying set $\{x\}$, $x \in X$, is in $\mathfrak{A}$.

Morphisms $f: (X, \mathfrak{A}) \to (Y, \mathfrak{B})$ are maps $f: X \to Y$ such that for each $A' \in \mathfrak{A}$ there exists $B' \in \mathfrak{B}$ for which $f: A' \to B'$ is a morphism in $\mathcal{K}$. (We denote domain-range restrictions of f by f again.)

Remark. We consider $\mathcal{K}$ as a subcategory of $A\mathcal{K}$, by identifying each object $A = (X, \alpha)$ of $\mathcal{K}$ with the object $(X, \mathfrak{A})$ of $A\mathcal{K}$ where

$$\mathfrak{A} = \{B \in \mathcal{K};\ B \sqsubseteq A\}.$$

4.3. Proposition. *For each top-category $\mathcal{K}$ with productive quotients, $A\mathcal{K}$ is a* CCT *extension of $\mathcal{K}$ in which $\mathcal{K}$ is finally dense.*

Remark. We can state explicitly what the hom-objects in $A\mathcal{K}$ are. Given objects $A = (X, \mathfrak{A})$ and $B = (Y, \mathfrak{B})$ in $A\mathcal{K}$, denote by $\mathcal{D}$ the set of all objects $D' = (T', \delta)$ in $\mathcal{K}$ such that

(a) $T' \subset \hom(A, B)$;

(b) for each $A' \in \mathfrak{A}$ there exists $B' \in \mathfrak{B}$ such that the evaluation map $\mathrm{ev}(g, x) = g(x)$ defines a morphism

$$\mathrm{ev}: D' \times A' \to B'$$

in $\mathcal{K}$.

We shall prove below that

$$\mathrm{Hom}(A, B) = (\mathrm{hom}(A, B), \mathscr{D})$$

is a canonical hom-object for A, B in $A\mathcal{K}$.

Proof. A) $A\mathcal{K}$ is a top-category. Indeed, $A\mathcal{K}$ is obviously well-fibred. For each source

$$\{X \xrightarrow{f_i} (Y_i, \mathfrak{B}_i)\}_{i \in I}$$

denote by $\mathfrak{A}$ the collection of all objects A' of $\mathcal{K}$ with the following properties:
- the underlying set of A' is a subset of X;
- for each $i \in I$ there exists $B_i' \in \mathfrak{B}_i$ such that $f_i : A' \to B_i'$ is a morphism in $\mathcal{K}$.

Then $\mathfrak{A}$ is the initial structure of the given source: it is easy to see that $(X, \mathfrak{A})$ is an object of $A\mathcal{K}$ and that $f_i : (X, \mathfrak{A}) \to (Y_i, \mathfrak{B}_i)$ are morphisms of $A\mathcal{K}$, for all $i \in I$. Thus, it suffices to show that for each object $(T, \mathscr{D})$ of $A\mathcal{K}$ and each map $h : T \to X$ such that all $f_i \cdot h$ are morphisms, $i \in I$, of $A'\mathcal{K}$ also h is a morphism, $h : (T, \mathscr{D}) \to (X, \mathfrak{A})$. For each $D' = (T', \delta)$ in $\mathscr{D}$ denote by A' the final object of the singleton sink $\{(T', \delta) \xrightarrow{h} X\}$. Then $h : D' \to A'$ is a morphism in $\mathcal{K}$; let us verify that $A' \in \mathfrak{A}$. We know that for each $i \in I$ there exists $B_i' \in \mathfrak{B}_i$ such that $f_i \cdot h : D' \to B_i'$ is a morphism in $\mathcal{K}$. It follows that $f_i : A' \to B_i'$ is a morphism in $\mathcal{K}$ (since A' is final), hence, $A' \in \mathfrak{A}$.

B) $A\mathcal{K}$ is cartesian closed, with hom-objects as described in the preceding remark. Indeed, the finite porduct of two objects $C = (Z, \mathfrak{C})$ and $A = (X, \mathfrak{A})$ is the following object

$$C \times A = (Z \times X, \mathfrak{C} \otimes \mathfrak{A})$$

where

$$\mathfrak{C} \otimes \mathfrak{A} = \{D \in \mathcal{K};\ D \sqsubseteq C' \times A' \text{ for some } C' \in \mathfrak{C},\ A' \in \mathfrak{A}\}.$$

This follows easily from the above description of initial structures. Now consider arbitrary objects

$$A = (X, \mathfrak{A}), \qquad B = (Y, \mathfrak{B}) \quad \text{and } C = (Z, \mathfrak{C})$$

and an arbitrary map $f : Z \times X \to Y$.

If $f : C \times A \to B$ is a morphism, we shall prove that $\hat{f} : C \to \mathrm{Hom}(A, B)$ is a morphism. For each $(C', \gamma) \in \mathfrak{C}$ put $D' = \hat{f}(C')$ and let (D', δ) be the final object of the singleton sink $\{(C', \gamma) \xrightarrow{h} D\}$; it suffices to prove that $(D', \delta) \in \mathscr{D}$. For each $(A', \alpha) \in \mathfrak{A}$ we have $(C', \gamma) \times (A', \alpha) \in \mathfrak{C} \otimes \mathfrak{A}$, hence, there exists $(B', \beta) \in \mathfrak{B}$ such that $f : (C', \gamma) \times (A', \alpha) \to (B', \beta)$ is a morphism. The following triangle

$$\begin{tikzcd}
(C', \gamma) \times (A', \alpha) \arrow[r, "f"] \arrow[dr, "f \times \mathrm{id}_A"'] & (B', \beta) \\
& (D', \delta) \times (A', \alpha) \arrow[u, "\mathrm{ev}"']
\end{tikzcd}$$

obviously commutes. Since $\hat{f}:(C', \gamma)\to(D', \delta)$ is a quotient in $\mathcal{K}$ and quotients in $\mathcal{K}$ are productive, we conclude that $\hat{f}\times\mathrm{id}_A$ is a quotient in $\mathcal{K}$. Hence, the fact that $\mathrm{ev}\cdot(\hat{f}\times\mathrm{id}_A)$ is a morphism implies that $\mathrm{ev}:(D', \delta)\times(A', \alpha)\to(B', \beta)$ is a morphism. This proves that $(D', \delta)\in\mathfrak{D}$.

Conversely, if $\hat{f}: C\to\mathrm{Hom}(A, B)$ is a morphism, we shall prove that $f: C\times A\to B$ is a morphism. For each $(T', \tau)\in\mathfrak{C}\otimes\mathfrak{A}$ there exists $(C', \gamma)\in\mathfrak{C}$ and $(A', \alpha)\in\mathfrak{A}$ with $(T', \tau)\sqsubseteq(C', \gamma)\times(A', \alpha)$; it suffices to show that there exists $(B', \beta)\in\mathfrak{B}$ such that $f:(C', \gamma)\times(A', \alpha)\to(B', \beta)$ is a morphism in $\mathcal{K}$. We know that for (C', γ) there exists $(D', \delta)\in\mathfrak{D}$ such that $\hat{f}:(C', \gamma)\to(D', \delta)$ is a morphism in $\mathcal{K}$. Since $(D', \delta)\in\mathfrak{D}$, for (A', α) there exists $(B', \beta)\in\mathfrak{B}$ such that ev: $(D', \delta)\times(A', \alpha)\to(B', \beta)$ is a morphism in $\mathcal{K}$. Finally, the triangle above commutes, hence, also $f:(C', \gamma)\times(A', \alpha)\to(B', \beta)$ is a morphism in $\mathcal{K}$.

C) The embedding of $\mathcal{K}$ into $A\mathcal{K}$ (Remark 4.2) is full, concrete and finally dense. Indeed, each object $(X, \mathfrak{A})$ is the final object of the following sink of inclusion maps:

$$\{(X', \alpha)\xrightarrow{\text{in}} X\}_{(X',\alpha)\in\mathfrak{A}}.$$

And any full, concrete and finally dense embedding of a top-category is easily seen to preserve finite products (in fact: all initial objects). Thus, $A\mathcal{K}$ is a CCT extension of $\mathcal{K}$. This concludes the proof.

Corollary. *Each top-category $\mathcal{K}$ with productive quotients has a* CCT *hull*: *the subcategory of $A\mathcal{K}$ formed by all initial objects of sources*

$$\{X\xrightarrow{f_i}\mathrm{Hom}(A_i, B_i)\}$$

with each A_i and B_i in $\mathcal{K}$.

4.3. Going a little further, assume that $\mathcal{K}$ has (besides productive quotients) a structure σ on each $\mathrm{hom}(A, B)$ with the following property (resembling that of the uniformity of uniform convergence, see 2.2(ii)):

> If (M, μ) is an object for which $M\subseteq\mathrm{hom}(A, B)$ and the evaluation map is a morphism, $\mathrm{ev}:(M, \mu)\to B$, then μ is finer than σ/M, and also $\mathrm{ev}:(M, \sigma/M)\to B$ is a morphism.

Then $\mathcal{K}$ has a smaller CCT extension than $A\mathcal{K}$. Denote by $A^*\mathcal{K}$ the category of all triples $(X, \alpha, \mathcal{A})$, where (X, α) is an object of $\mathcal{K}$ and $\mathcal{A}\subseteq\exp X$ such that α is final with respect to the sink of inclusion maps $\{(P, \alpha/P)\to X\}$ for all $P\in\mathcal{A}$. Morphisms $f:(X, \alpha, \mathcal{A})\to(Y, \beta, \mathcal{B})$ are those morphisms $f:(X, \alpha)\to(Y, \beta)$ in $\mathcal{K}$ which fulfil $f(P)\in\mathcal{B}$ for each $P\in\mathcal{A}$.

$A^*\mathcal{K}$ is a CCT extension of $\mathcal{K}$. The hom-object $\mathrm{Hom}(A, B)$ of two objects of $\mathcal{K}$ is

$$(\mathrm{hom}(A, B), \sigma_0, \mathcal{S}),$$

where $\mathcal{S}$ is the set of all $M \subseteq \hom(A, B)$ for which $\mathrm{ev}: (M, \sigma/M) \to B$ is a morphism, and σ_0 is the $\mathcal{S}$-final modification of σ.

If $\mathcal{X}$ has, in addition, productive finite coproducts, then the full subcategory of $A^*\mathcal{X}$ consisting of all $(X, \alpha, \mathcal{A})$ for which $\mathcal{A}$ is a bornology is a CCT extension, too.

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